Estimating Exponential Affine Models With Correlated Measurement Errors:Applications to Fixed Income and Commodities*

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Abstract

Exponential affine models (EAMs) are factor models popular in financial asset pricing requiring a dynamic term structure, such as for interest rates and commodity futures. When implementing EAMs it is usual to first specify the model in state space form (SSF) and then to estimate it using the Kalman filter. To specify the SSF, a structure of the measurement error must be provided which is not specified in the EAM itself. Different specifications of the measurement errors will result in different SSFs, leading to different parameter estimates. In this paper we investigate the influence of the measurement error specification on the parameter estimates. Using market data for both fixed income and commodities we provide evidence that measurement errors are cross-sectionally and serially correlated, which is not consistent with the independent identically distributed (iid) assumptions commonly adopted in the literature. Using simulated data we show that measurement error assumptions affect parameter estimates, especially in the presence of serial correlation. We provide a new specification, the augmented state space form (ASSF), as a solution to these biases and show that the ASSF gives much better estimates than the basic SSF.

Keywords: exponential affine model, state space form, Kalman filter, EM algorithm, measurement errors, serial correlation, commodity futures, yield curves

JEL Codes: G12, G13

1 Introduction

Exponential affine models (EAMs) are important, and commonly-used, models in the asset pricing arena, especially in the term-structure modeling of fixed income securities and commodity futures. EAMs model financial instruments such as zero-coupon bonds and commodity futures as a function of latent factors and time-to-maturity. There are many well-known studies on financial instruments using EAMs. For the literature on interest rate term-structure modeling and estimation, we refer the reader to Chen and Scott (1993), De Munnik and Schotman (1994), Duan and Simonato (1999), De Jong and Santa-Clara (1999), De Jong (2000), Dai and Singleton (2000), Zhou (2001), Chen and Scott (2003) and Christensen *et al.* (2007). For the commodity literature, see Schwartz (1997), Schwartz and Smith (2000), Geman and Nguyen (2005), Casassus and Collin-Dufresne (2005) and Dempster, Medova and Tang (2008, 2009).

The main difficulty involved in estimating EAMs is that their factors (or state variables) are *latent*, i.e. not directly observable. Several calibration methods have been proposed to solve this problem, such as the use of *proxies* for the latent factors (Marsh and Rosenfeld, 1983 and Daves and Ehrhardt, 1993), the *efficient method of moments* (EMM) (Gallant and Tauchen, 1996) and the Chen and Scott (1993) method. In the context of EAMs, the logarithms of financial instrument prices are specified using an affine function of the latent variables. For models with Gaussian factors, the *Kalman filter* is appropriate for the estimation of EAM parameters and is commonly believed to be the best estimator for linear Gaussian models. For models with non-Gaussian factors of Cox-Ingersoll-Ross (CIR) type, the *extended* Kalman filter is often used for model calibration. Duffee and Stanton (2004) compare several parameter estimation methods and conclude that the (extended) Kalman filter is the best for such non-Gaussian models. We refer readers to Duan and Simonato (1999), De Jong (2000) and Duffee and Stanton (2004) for the application of the extended Kalman filter to interest rate modeling. De Jong (2000) calibrates interest rate models using one, two and three non-Gaussian latent factors. Using simulated data in which measurement errors are cross-sectionally and serially *uncorrelated*, he finds that the Kalman filter gives quite precise parameter estimates. Note that before estimating EAMs using the Kalman filter, an EAM must first be written in *state-space form* (SSF).

When the number of contracts is the same as the number of latent factors, the factors can be identified with the contracts which can be priced exactly (Chen and Scott, 1993 and Casassus and Collin-Dufresne, 2005). However, nearly all of the cited papers use more market-observed contracts than latent variables, and thus a specification of *measurement error* is necessary. However, an EAM itself is not sufficient to determine its state-space formulation because it does not address the measurement error *structure* which must be specified in order to implement a model. In other words, an EAM together with the specification of an error structure is required to estimate the specified model. Different formulations of measurement errors result in different state space forms and, hence, in different parameter estimates. To the best of our knowledge, for all term structure papers that use the Kalman filter for model calibration, such as Duan and Simonato (1999), De Jong (2000), Chen and Scott (2003) in the fixed income literature and Schwartz (1997), Schwartz and Smith (2000), Geman and Nguyen (2005) in the commodity literature, the measurement errors are assumed for computational convenience to be *independently and identically distributed* (iid) and their covariance matrix is commonly assumed to be a *diagonal* matrix.[1] However, when calibrating interest rates models using one, two and three factor models, De Jong (2000) finds that both strong cross-sectional and serial correlations exist in the measurement errors. Unfortunately, he does not provide a solution to the estimation problem. Figures 1, 2 and 3 show that strong cross-sectional and serial correlations exist in measurement errors when using the one, two and three factor models of Appendix C applied to oil futures. Thus, the iid assumption for measurement errors is violated in term structure modeling for both interest rates and commodity futures. It is of course not surprising that the movements of interest rates or futures prices of similar maturities are highly correlated, both contemporaneously and individually over time, with higher autocorrelation at higher observation frequencies. However, until now, no research has investigated how this inconsistency influences parameter estimates and, more importantly, how the resulting biases can be resolved. In this paper, we aim to fill this gap by first showing that any

[1]In many cases, to reduce computational complexity, the standard deviations of the measurement errors of different contracts are assumed to be identical; see Sorensen (2002).

such inconsistency *does* affect parameter estimation and then proposing a solution (that is, a new SSF) to alleviate the problem. We emphasize commodity futures models in this paper because the EAMs of commodity futures are relatively new and require further understanding. Moreover, nearly all commodity models in the literature have been calibrated using the Kalman filter.

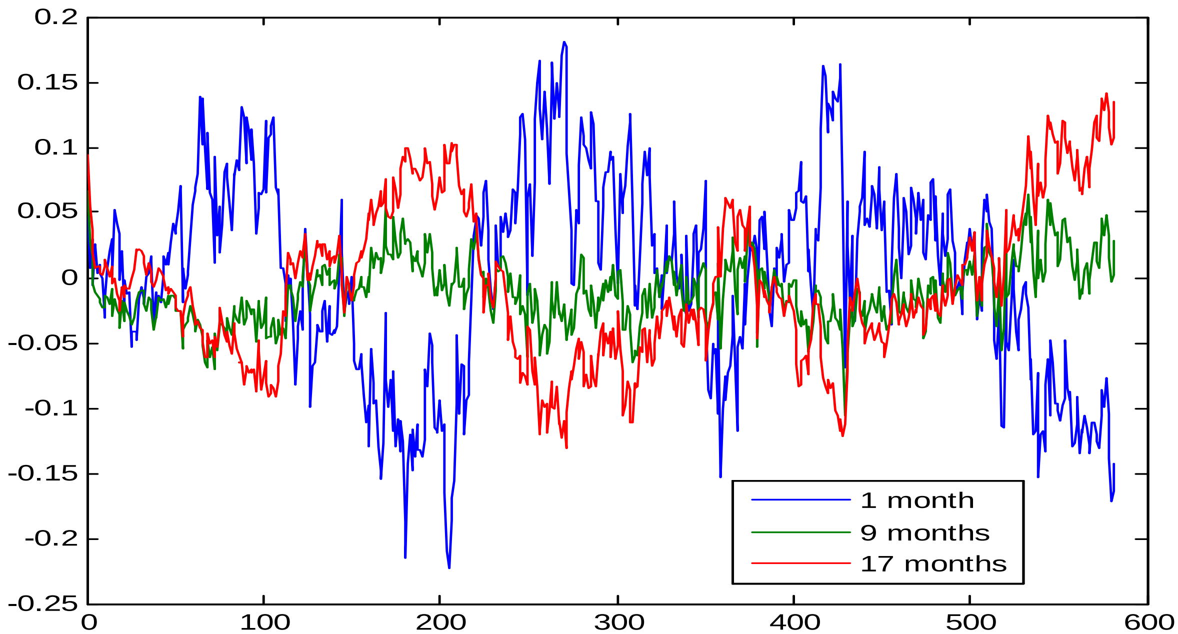

Figure 1: **Measurement errors of the one-factor model applied to oil futures**

This paper is organized as follows. Section 2 describes the structure and implementation of EAMs. Section 3 investigates cross-sectional and serial correlation in measurement errors using both simulated and actual market data. Section 4 provides a solution to overcome the iid assumptions violation and Section 5 concludes.

2 State-Space Formulation of the Exponential Affine Model

In this section we introduce the EAM and its state-space formulation.

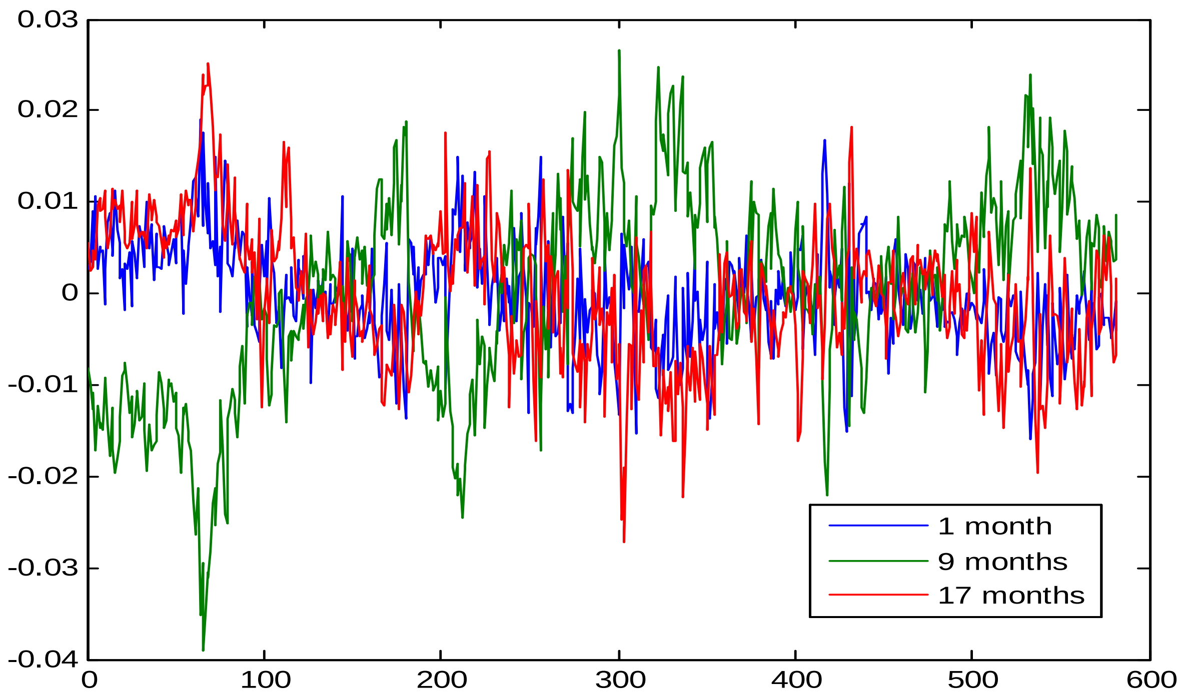

Figure 2: **Measurement errors of the two-factor model applied to oil futures**

2.1 The exponential affine model

We define $\boldsymbol{x}_t$ as the driving variable underlying the interest rate or commodity futures term structure, which for example could be a short interest rate $\boldsymbol{r}_t$ or the log spot price of a commodity $\boldsymbol{v}_t$. Letting t, T and $\tau := T - t$ denote the *current time*, *contract maturity* and *time to maturity*, respectively, one can obtain the *futures price* of the commodity[2] $F_t(\tau)$ and *zero coupon bond price* $P_t(\tau)$ respectively as [3]

$$F_t(\tau) = \mathbb{E}_t^Q[\exp(\boldsymbol{v}_T)], \tag{1}$$

and

$$P_t(\tau) = \mathbb{E}_t^Q[\exp(-\int_t^T \boldsymbol{r}_s ds)], \tag{2}$$

[2]Note that $F_t(\tau)$ is often written as $F(t, T)$ in the commodity literature.

[3]Throughout the paper we use boldface to denote random or conditionally random entities.

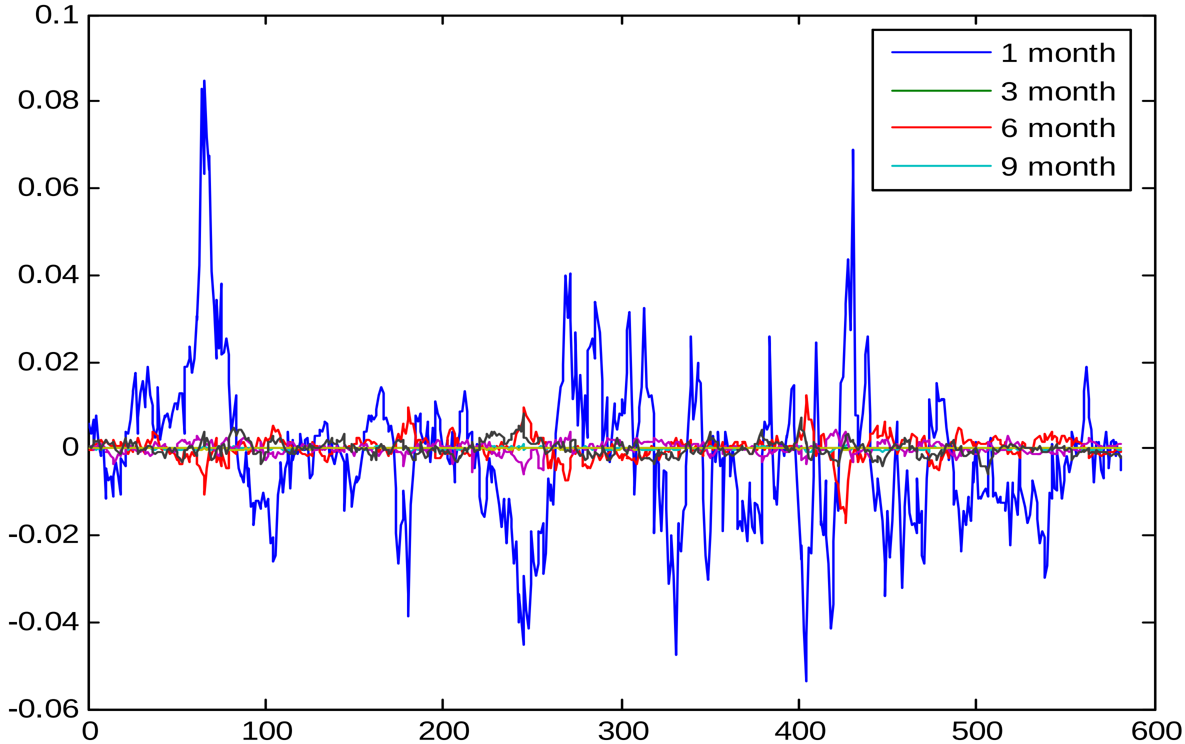

Figure 3: **Measurement errors of the three-factor model applied to oil futures**

where $\boldsymbol{v}_t$ denotes the logarithm of a *commodity spot price*, $\boldsymbol{r}_t$ denotes an *instantaneous short interest rate* and the conditional expectations at t are taken under the risk-neutral measure Q.

We follow Duffie and Kan (1996), Duffie, Pan and Singleton (2000), and Dai and Singleton (2000) in introducing the canonical representation of an N-factor state vector $\boldsymbol{Y}_t$ driving the movement of $\boldsymbol{x}_t$ ($\boldsymbol{v}_t$ or $\boldsymbol{r}_t$) as

$$\boldsymbol{x}_t = \phi_0 + {\phi_Y}' \boldsymbol{Y}_t,$$

where ϕ_0 is a constant, ϕ_Y is an $N \times 1$ vector, and $\boldsymbol{Y}_t = (\boldsymbol{Y}_{1,t}, \boldsymbol{Y}_{2,t}, \ldots, \boldsymbol{Y}_{N,t})'$ is a vector of N *state variables*. These variables follow a square root diffusion process under the risk-neutral measure Q satisfying

$$d\boldsymbol{Y}_t = K(\Theta - Y_t)dt + \Sigma\sqrt{\Upsilon_t}d\boldsymbol{W}_t^Q, \tag{3}$$

where K and Σ are both $N \times N$ constant matrices, Θ is an $N \times 1$ vector and $\mathbf{W}_t^Q$ is an $N \times 1$ vector of independent Brownian motions. Conditional on $\mathbf{Y}_t$, Υ_t is a diagonal matrix with the i^{th} diagonal element given by

$$\Upsilon_{t,ii} = \delta_i + \gamma_i' Y_t. \tag{4}$$

Assuming that the parameterization is admissible, it is easy to show that given Y_t both $F_t(\tau)$ and $P_t(\tau)$ can be obtained in the form

$$Z_t(\tau) = \exp[A(\tau) + B'(\tau)Y_t], \tag{5}$$

where $Z_t(\tau)$ represents either $F_t(\tau)$ or $P_t(\tau)$, $A(\tau)$ is a constant and $B(\tau)$ is an $N \times 1$ vector, all depending on τ. Because the *price* $Z_t(\tau)$ can be written in terms of the latent factors Y_t in an exponential affine form, a model of this kind is termed an *exponential affine model* (EAM). Defining $z_t(\tau) := \ln(Z_t(\tau))$, the new variable has the affine structure

$$z_t(\tau) = A(\tau) + B'(\tau)Y_t. \tag{6}$$

Letting $U := (\gamma_1, \ldots, \gamma_N)$ denote the matrix of coefficients of Y in Υ and defining $m := rank(U)$, $(m \leq N)$ indexes the degree of dependence of the conditional covariance on the state variables. Dai and Singleton (2000) classify each EAM into one of $N+1$ subfamilies $\mathbb{A}_m(N)$ based on the value of m.

In the market (physical) measure P (i.e., the data-generating measure) the *risk premium* process $\mathbf{\Lambda}_t$ is specified through

$$d\mathbf{W}_t^Q = \Lambda_t dt + d\mathbf{W}_t^P, \tag{7}$$

where $\Lambda_t := \sqrt{\Upsilon}\lambda$ (given Y_t) and λ is an $N \times 1$ vector of constants. Thus in the market measure (3) becomes

$$d\mathbf{Y}_t = \widetilde{K}(\widetilde{\Theta} - Y_t)dt + \Sigma\sqrt{\Upsilon}d\mathbf{W}_t^P, \tag{8}$$

where $\widetilde{K} = K - \Sigma U, \widetilde{\Theta} = \widetilde{K}^{-1}(K\Theta + \Sigma\psi)$, the i^{th} row of the $N \times N$ matrix U is given by $\lambda_i \gamma_i'$ and ψ is an $N \times 1$ vector with i^{th} element given by $\lambda_i \delta_i$. Note that this specification of the risk premia allows the factor process $\boldsymbol{Y}_t$ also to follow the affine form of (3) in the market measure.

Given the factor processes in the risk-neutral and market measures, to estimate the parameters of the EAM from market data we must first specify the measurement error structure in the *state space form* (SSF) of the model and then use the Kalman filter in estimating parameters. In the next section, we discuss a basic SSF of the EAM.

2.2 Basic state space form

The SSF normally consists of a transition equation and a measurement equation in discrete time. The transition equation describes the evolution of the stochastic process of the latent state variables in the market measure. The measurement equation relates a multivariate time series of observable variables to the latent (unobservable) state variables and is commonly related to the risk-neutral process. In this section, we discuss in detail the basic state space form which comes directly from the factor process under the market and risk-neutral measures.

The *transition equation* is given by discretizing (8) to yield

$$\boldsymbol{Y}_t = d + \Phi Y_{t-1} + \boldsymbol{\eta}_t, \tag{9}$$

where d is a $N \times 1$ vector, Φ is a $N \times N$ matrix and $\boldsymbol{\eta}_t$ is a $N \times 1$ normally distributed random vector with mean 0 and covariance matrix Ω_t. The parameters of (9) are derived from the conditional mean and variance of the factors as

$$\begin{aligned} E[\boldsymbol{Y}_t | Y_{t-1}] &= d + \Phi Y_{t-1} \\ var(\boldsymbol{\eta}_t) &= var(\boldsymbol{Y}_t | Y_{t-1}) = \Omega(Y_{t-1}) := \Omega_t. \end{aligned}$$

De Jong (2000) has shown that both $E[\boldsymbol{Y}_t | Y_{t-1}]$ and $var(\boldsymbol{\eta}_t)$ are affine functions of Y_{t-1}. It is obvious that for $\mathbb{A}_0(N)$ models, Ω is a constant and does not depend on Y_{t-1}. In this case, the Kalman filter will lead to unbiased estimates for the EAM parameters. For $\mathbb{A}_{m\neq 0}(N)$ models, if the latent factors at $t-1$ are known exactly, $\Omega(Y_{t-1})$ will be a known function of the parameters and the Kalman filter will again generate unbiased estimates. However, because the filtered estimate $\widehat{Y}_{t-1}$ must be used instead of the (unknown) true value Y_{t-1} in the Kalman filter iteration, there will be some bias. Nevertheless, as suggested by Duan and Simonato (1999) and De Jong (2000), the classical Kalman filter recursions still give reasonable parameter estimates with the specified measurement errors.

Another important characteristic of the covariance matrix Ω_t is its positive semidefiniteness. However, this feature is not guaranteed in the discretized process in (9) because the CIR-type factors given by (8) can not be guaranteed to be nonnegative in the Kalman filter iterations.[4] We thus follow Chen and Scott (2003) in replacing the negative estimates with zeros.[5] Note that we use the extended Kalman filter (see e.g. De Jong 2000) to calibrate the $\mathbb{A}_{1,DS}(3)$ and $\mathbb{A}_{2,DS}(3)$ CIR-type models. After obtaining the model parameters and latent factors, we check the nonnegativity of the corresponding CIR-type factor estimates, i.e. of the volatility factor v in the $\mathbb{A}_{1,DS}(3)$ model, and the long-run mean factor θ and volatility factor v in the $\mathbb{A}_{2,DS}(3)$ model. It turns out that all CIR-type factor estimates are positive.

The measurement equation (6) shows that an exact relation exists between the latent factors Y_t and the logarithm of the observed price $z_t(\tau)$. However, when using more observed price maturities than factors, this relationship is overdetermined and cannot be satisfied by all log prices. Thus the specification of a *measurement error process* is necessary to determine the measurement equation. Observe that the inclusion of measurement errors also aims at capturing factors which are not explicitly accounted for in the model, including, for example, sampling errors, bid-ask spreads and non-simultaneity of the observations. A natural specification is to add to (6) iid errors ε_t which are usually assumed to be Gaussian. For non-Gaussian measurement errors we refer the reader to, for example, Shephard (1994) for a treatment of t-distributed errors.

[4] We thank the referee for pointing this out.

[5] Chen and Scott (2003) show by simulation, that this approximation works well in estimating CIR-type factors.

Therefore, assuming there are M $(M > N)$ observations at time t, the *measurement equation* is specified as

$$\boldsymbol{\mathcal{Z}}_t = \mathcal{A} + \mathcal{B}Y_t + \boldsymbol{\varepsilon}_t \tag{10}$$

where $\mathcal{Z}_t = (z_t(\tau_1), ..., z_t(\tau_M))'$ and $\mathcal{A} = (A(\tau_1), ..., A(\tau_M))'$ are $M \times 1$ vectors and $\mathcal{B} = (B(\tau), ..., B(\tau))'$ is an $M \times N$ matrix. Note that here $\boldsymbol{\varepsilon}_t$ is a measurement error random vector which is joint normally distributed with mean zero and covariance matrix R. A standard assumption is that $\boldsymbol{\eta}_t$ in the transition equation and $\boldsymbol{\varepsilon}_t$ in the measurement equation are *independent*. Since, given the EAM, this specification of the SSF is relatively simple and straightforward, we term the state space form given by (9) and (10) the *basic* SSF in order to distinguish it from the augmented SSF proposed below. The basic SSF is employed in all papers that use the Kalman filter for parameter estimation. Appendix A sets out the Kalman filter estimation algorithm adapted from Harvey (1989). Before investigating in Section 3 how the specification of the measurement error process affects the model parameter estimates, we briefly examine in the next section the number of latent factors required to specify an EAM.

2.3 Are three factors sufficient?

Many researchers such as Schwartz (1997) and de Jong (2000) have raised concerns that cross-sectional and serial measurement error correlations may be caused by an incomplete model in which the underlying dynamics are not fully captured. In order to test how many factors are enough to model a term-structure dataset, *principal component analysis* (PCA) is the method commonly-used. In this section we use PCA on the oil futures data (i.e., 1, 3, 6, 9, 12, 15, 17 month contracts) and the interest rate term-structure data (i.e., 3, 6, 12, 24, 36, 48, 60, 84, 120 months rates) of Section 3.[6] Table 1 shows the results. We see that three factors explain more than 99.9% of the variance of both interest rate and oil futures movements in our data. This is consistent with several studies, such as Litterman and Scheinkman (1991) and Cortazar

[6]Because PCA must be conducted on stationary data, we use here returns of futures contracts in the analysis. Interest rates are commonly considered to be mean-reverting; thus we use the interest rates themselves in the analysis, which is consistent with Litterman and Scheinkman (1991).

Table 1: **PCA results for the oil futures and interest rates**

	variance explained – interest rates (%)	variance explained – oil futures (%)
first factor	95.70	93.55
second factor	4.06	5.89
third factor	0.21	0.47
fourth factor	0.02	0.07
fifth factor	0.00	0.02

and Schwartz (1994). Moreover, using more than three factors may cause over-fitting. However, as shown in Section 3.1, even for the three-factor interest rate and commodity futures models, measurement errors still exhibit strong contemporaneous cross-sectional and serial correlations. We may conclude that the usual iid assumptions for measurement errors are *not* appropriate for EAMs. Specific measurement error treatments, such as that of the augmented SSF used in this paper, are therefore necessary.

3 Measurement Error Cross-sectional and Serial Correlation

In this section, we first test whether measurement errors are cross-sectionally and serially correlated using actual market data and then using simulated data study how cross-sectional and serial correlations affect model parameter estimates.

3.1 Test of iid measurement errors

Using one, two and three factor models, De Jong (2000) has shown that strong cross-sectional and serial correlations exist in the estimated measurement errors in interest rate term-structures.

Fixed Income

Here we confirm these results using the three models set out in Appendix B. The first is a three factor generalized Vasicek term structure model with which we have had considerable experience for both asset-liability management (Dempster *et al.*, 2006, 2007) and pricing (Dempster *et al.*, 2009) over long horizons using daily, weekly or monthly data as appropriate to the application.

The second and third models are the $\mathbb{A}_{1,DS}(3)$ and $\mathbb{A}_{2,DS}(3)$ of Dai and Singleton (2000), which they show are the preferable $\mathbb{A}_1(3)$ and $\mathbb{A}_2(3)$ models, respectively.[7]

To estimate these models, we use weekly interest rate data from the Federal Reserve (1987.07 to 2007.07, 1045 observations) with 3, 6, 12, 24, 36, 48, 60, 84, 120 months maturities. We first obtain zero-coupon bond yields from the interest rates by a boot-strapping method, using the fact that zero-coupon bond *yield* $p_t(\tau)$ is calculated by taking the logarithm of the zero price, i.e. $p_t(\tau) := -\frac{\ln(P_t(\tau))}{\tau}$. We employ the basic SSF given by (9) and (10) and use the Kalman filter in the EM algorithm to estimate parameters (see Appendix A). The standard deviation of measurement errors for different contracts are assumed to be different, i.e. the covariance matrix of the $\boldsymbol{\varepsilon}_t$s is specified as $R := diag\,(\sigma_1^2, ..., \sigma_9^2)$, where $\sigma_1, ..., \sigma_9$ are constant volatilities for the nine maturities.

An obvious way to check whether measurement errors violate the iid assumptions is to explore the statistics of their residual estimates $\widehat{\varepsilon}_t$ (Harvey, 1989). Tables 2, 3 and 4 show the statistics of the measurement errors, which show strong serial and cross-sectional correlations.[8] Next we examine another important application of EAMs, namely, commodity futures.

Commodities

Schwartz (1997) and Gibson and Schwartz (1990) provide one and two factor models and Dempster, Medova and Tang (2009) present a three factor model.[9] Appendix C sets out these models. Since all three belong to the $\mathbb{A}_0(N)$ $(N = 1, 2, 3)$ family of EAMs, the Kalman filter leads to unbiased estimates under the iid assumptions on the measurement errors. We will test these assumptions using the one, two and three factor models in the context of the oil markets.

We use weekly oil futures prices from the New York Mercantile Exchange (NYMEX) (1995.01 to 2006.02, 582 observations) with 1, 9, 17 month maturities to estimate the one- and two-factor

[7] We thank the referee for suggesting modeling $\mathbb{A}_1(3)$ and $\mathbb{A}_2(3)$ type models.

[8] For future comparison we also present here (and below) the results for the new state space form of model discussed in Section 4.2.

[9] Note that Schwartz (1997) and Casassus and Collin-Dufresne (2005) also present three-factor models with the third factor being stochastic interest rates. Because estimating their models involves information not only from the commodity futures market but also from the fixed-income bond market, these models involve additional complexities and thus neither is used here. Instead, we use the Dempster, Medova and Tang (2009) model as the representative three-factor model.

Table 2: **Statistics of residuals $\widehat{\varepsilon}_t$ in the basic SSF and $\widehat{u}_t$ in the augmented SSF for the three factor generalized Vasicek interest rate model**

	Covariance matrix									DW	Serial correlation
Basic SSF											
3 months	1.00	-0.62	-0.66	-0.40	0.26	0.30	0.12	-0.31	-0.37	**0.33**	**0.84** (0.017)
6 months	-0.62	1.00	0.17	-0.16	-0.20	0.06	0.12	0.06	0.05	**1.42**	**0.29** (0.029)
12 months	-0.66	0.10	1.00	0.58	-0.39	-0.31	-0.14	0.39	0.37	**0.54**	**0.73** (0.021)
24 months	-0.40	-0.16	0.58	1.00	-0.13	-0.56	-0.37	0.50	0.53	**0.37**	**0.82** (0.018)
36 months	0.26	-0.28	-0.39	-0.13	1.00	-0.53	-0.51	0.04	0.27	**1.27**	**0.37** (0.028)
48 months	0.31	0.06	-0.31	-0.56	-0.53	1.00	0.45	-0.59	-0.62	**0.83**	**0.58** (0.025)
60 months	0.12	0.12	-0.14	-0.37	-0.51	0.45	1.00	-0.58	-0.58	**0.49**	**0.75** (0.020)
72 months	-0.31	0.06	0.39	0.50	0.04	-0.59	-0.58	1.00	0.33	**0.67**	**0.67** (0.023)
120 months	-0.37	0.05	0.37	0.53	0.27	-0.62	-0.58	0.33	1.00	**0.29**	**0.86** (0.016)
ASSF											
3 months	1.00	0.93	0.98	0.91	0.95	0.91	0.91	0.95	0.95	2.37	**-0.19**(0.030)
6 months	0.93	1.00	0.93	0.85	0.92	0.92	0.92	0.93	0.88	2.35	**-0.18**(0.031)
12 months	0.98	0.93	1.00	0.83	0.94	0.94	0.93	0.93	0.88	**2.59**	**-0.29**(0.030)
24 months	0.91	0.85	0.83	1.00	0.84	0.74	0.79	0.90	0.94	2.02	-0.01(0.030)
36 months	0.95	0.92	0.94	0.84	1.00	0.85	0.84	0.91	0.93	2.47	**-0.24**(0.030)
48 months	0.91	0.92	0.94	0.74	0.85	1.00	0.86	0.86	0.82	**2.59**	**-0.30** (0.031)
60 months	0.91	0.92	0.93	0.79	0.84	0.86	1.00	0.87	0.79	2.49	**-0.25**(0.030)
72 months	0.95	0.93	0.93	0.90	0.91	0.86	0.87	1.00	0.87	2.38	**-0.19**(0.030)
120 months	0.95	0.88	0.88	0.94	0.93	0.82	0.79	0.87	1.00	2.17	**-0.09**(0.030)

Note that the Durbin-Watson (DW) statistic is about 1.43 for 1% significant level. The estimates of serial correlation $\widehat{\rho}$ are obtained from estimating $\boldsymbol{\varepsilon}_t = \rho\varepsilon_{t-1} + \boldsymbol{e}_t$ for each maturity and the quantities in the brackets beside them are their estimated standard deviations. Boldface denotes significant at the 1% level.

Table 3: **Statistics of residuals $\widehat{\varepsilon}_t$ in the basic SSF and $\widehat{u}_t$ in the augmented SSF for the three factor $A_{1,DS}(3)$ interest rate model**

	Covariance matrix									DW	Serial correlation
Basic SSF											
3 months	1.00	0.95	0.50	-0.50	-0.46	0.52	0.50	-0.03	-0.58	**0.14**	**0.93** (0.011)
6 months	0.95	1.00	0.69	-0.36	-0.59	0.47	0.49	0.10	-0.60	**0.26**	**0.87** (0.015)
12 months	0.50	0.69	1.00	0.19	-0.75	0.15	0.24	0.42	-0.37	**0.98**	**0.51** (0.026)
24 months	-0.50	-0.36	0.19	1.00	-0.29	-0.51	-0.41	0.58	0.31	**0.50**	**0.75** (0.020)
36 months	-0.46	-0.59	-0.75	-0.29	1.00	-0.52	-0.54	-0.16	0.51	**0.80**	**0.60** (0.024)
48 months	0.52	0.47	0.15	-0.51	-0.52	1.00	0.71	-0.59	-0.49	**0.72**	**0.64** (0.023)
60 months	0.50	0.49	0.24	-0.41	-0.54	0.71	1.00	-0.52	-0.57	**0.34**	**0.82** (0.017)
72 months	-0.03	0.10	0.42	0.58	-0.16	-0.59	-0.52	1.00	-0.17	**0.79**	**0.60** (0.024)
120 months	-0.58	-0.60	-0.37	0.31	0.51	-0.49	-0.57	-0.17	1.00	**0.19**	**0.90** (0.013)
ASSF											
3 months	1.00	0.90	0.73	-0.10	-0.12	0.12	0.21	0.41	0.38	2.23	**-0.12** (0.030)
6 months	0.90	1.00	0.88	-0.02	-0.06	0.16	0.25	0.46	0.44	2.46	**-0.23** (0.030)
12 months	0.73	0.88	1.00	0.09	0.06	0.19	0.29	0.46	0.45	**2.77**	**-0.39** (0.028)
24 months	-0.10	-0.02	0.09	1.00	0.84	0.85	0.73	0.76	0.82	2.25	**-0.13** (0.029)
36 months	-0.12	-0.06	0.06	0.84	1.00	0.78	0.58	0.65	0.73	2.44	**-0.22** (0.029)
48 months	0.12	0.16	0.19	0.85	0.78	1.00	0.43	0.65	0.78	2.37	**-0.18** (0.030)
60 months	0.21	0.25	0.29	0.73	0.58	0.43	1.00	0.81	0.85	2.42	**-0.21** (0.029)
72 months	0.41	0.46	0.46	0.76	0.65	0.65	0.81	1.00	0.85	2.37	**-0.19** (0.030)
120 months	0.38	0.44	0.45	0.82	0.73	0.78	0.85	0.85	1.00	2.24	**-0.12** (0.030)

Table 4: **Statistics of residuals $\widehat{\varepsilon}_t$ in the basic SSF and $\widehat{u}_t$ in the augmented SSF for the three factor $A_{2,DS}(3)$ interest rate model**

	Covariance matrix									DW	Serial correlation
Basic SSF											
3 months	1.00	0.97	0.69	-0.18	-0.52	0.27	0.37	0.15	-0.43	**0.11**	**0.94** (0.010)
6 months	0.97	1.00	0.82	-0.04	-0.60	0.19	0.36	0.24	-0.45	**0.19**	**0.91** (0.013)
12 months	0.69	0.82	1.00	0.34	-0.66	-0.03	0.22	0.41	-0.39	**0.64**	**0.68** (0.022)
24 months	-0.18	-0.04	0.34	1.00	-0.31	-0.52	-0.27	0.52	0.02	**0.53**	**0.73** (0.021)
36 months	-0.52	-0.60	-0.66	-0.31	1.00	-0.48	-0.54	-0.17	0.54	**0.70**	**0.65** (0.023)
48 months	0.27	0.19	-0.03	-0.52	-0.48	1.00	0.46	-0.50	-0.23	**1.00**	**0.50** (0.026)
60 months	0.37	0.36	0.22	-0.27	-0.54	0.46	1.00	-0.42	-0.27	**0.51**	**0.73** (0.020)
72 months	0.15	0.24	0.41	0.52	-0.17	-0.50	-0.42	1.00	-0.59	**0.60**	**0.70** (0.022)
120 months	-0.43	-0.45	-0.39	0.02	0.54	-0.23	-0.27	-0.59	1.00	**0.33**	**0.84** (0.017)
ASSF											
3 months	1.00	0.95	0.81	0.37	0.53	0.39	0.29	0.31	0.21	2.16	**-0.08** (0.030)
6 months	0.95	1.00	0.91	0.39	0.54	0.39	0.30	0.37	0.26	2.38	**-0.19** (0.030)
12 months	0.81	0.91	1.00	0.34	0.48	0.32	0.30	0.41	0.33	**2.78**	**-0.29** (0.028)
24 months	0.37	0.39	0.34	1.00	0.91	0.63	0.54	0.49	0.30	**2.64**	**-0.32** (0.029)
36 months	0.53	0.54	0.48	0.91	1.00	0.88	0.41	0.22	0.05	**2.74**	**-0.37** (0.028)
48 months	0.39	0.39	0.32	0.63	0.88	1.00	0.12	-0.23	-0.34	2.32	**-0.31** (0.028)
60 months	0.29	0.30	0.30	0.54	0.41	0.12	1.00	0.41	0.26	**2.64**	**-0.32** (0.029)
72 months	0.31	0.37	0.41	0.49	0.22	-0.23	0.41	1.00	0.70	2.43	**-0.26** (0.029)
120 months	0.21	0.26	0.33	0.30	0.05	-0.34	0.26	0.70	1.00	2.18	**-0.09** (0.030)

Table 5: **Statistics of residuals $\widehat{\varepsilon}_t$ in the basic SSF and $\widehat{u}_t$ in the augmented SSF for the one factor commodity model**

	Covariance matrix			DW	Serial Correlation	
Basic SSF						
1 month	1.00	-0.39	-0.81	**0.19**	**0.90**	(0.018)
9 month	-0.39	1.00	0.76	**0.40**	**0.80**	(0.024)
17 month	-0.81	0.76	1.00	**0.06**	**0.97**	(0.011)
ASSF						
1 month	1.00	0.79	0.81	1.65	**0.17**	(0.040)
9 month	0.79	1.00	0.99	**1.40**	**0.30**	(0.039)
17 month	0.81	0.99	1.00	**1.39**	**0.30**	(0.039)

Note that the Durbin-Watson (DW) statistic is about 1.43 for 1% significant level. The estimates of serial correlation $\widehat{\rho}$ are obtained from estimating $\boldsymbol{\varepsilon}_t = \rho\varepsilon_{t-1} + \boldsymbol{e}_t$ for each maturity and the quantities in the brackets beside them are their estimated standard deviations. Boldface denotes significant at the 1% level.

Table 6: **Statistics of residuals $\widehat{\varepsilon}_t$ in the basic SSF and $\widehat{u}_t$ in the augmented SSF for the two factor commodity model**

	Covariance matrix			DW	Serial correlation	
Basic SSF						
1 month	1.00	-0.79	0.25	**0.92**	**0.54**	(0.034)
9 months	-0.79	1.00	-0.64	**0.22**	**0.89**	(0.018)
17 months	0.25	-0.64	1.00	**0.48**	**0.76**	(0.026)
ASSF						
1 month	1.00	-0.99	-0.91	2.04	-0.02	(0.040)
9 months	-0.99	1.00	0.90	2.03	-0.02	(0.041)
17 months	-0.91	0.90	1.00	2.06	-0.03	(0.040)

models, and we use 1, 3, 6, 9, 12, 15, 17 months futures to estimate the three-factor models.

By assuming identical standard deviations for all contracts, we specify the covariance matrix of measurement errors as $R := \sigma_\varepsilon^2 I_3$, where I_M denotes $M \times M$ identity matrix, for the one and two factor models. For the three factor model we allow different standard deviations for each contract; i.e., $R = diag(\sigma_1^2, ..., \sigma_7^2)$. We report the measurement errors for the basic and our augmented SSF in this section and the corresponding parameter estimates in Section 4.1.

Figures 1 to 3 in Section 1 show estimated measurement error series for the one, two and three factor models. Tables 5 to 7 show the summary statistics of these residual estimates $\widehat{\varepsilon}_t$ for the three models. From the correlation matrices and DW statistics, we see that strong cross-sectional and serial correlation exists in the residuals, which violates the iid assumption of the basic SSF.

Table 7: **Statistics of residual $\widehat{\varepsilon}_t$ in the basic SSF and $\widehat{u}_t$ in the augmented SSF for the three factor model**

	Covariance matrix							DW	Serial correlation	
Basic SSF										
1 month	1.00	-0.20	-0.59	0.19	0.29	-0.13	-0.08	**0.27**	**0.87**	(0.020)
3 months	-0.20	1.00	-0.29	-0.59	0.32	0.33	-0.34	**1.48**	**0.26**	(0.039)
6 months	-0.59	-0.29	1.00	-0.27	-0.54	0.31	0.08	**0.32**	**0.84**	(0.022)
9 months	0.19	-0.59	-0.27	1.00	-0.39	-0.63	0.61	**1.05**	**0.48**	(0.036)
12 months	0.29	0.32	-0.54	-0.39	1.00	-0.17	-0.38	**0.53**	**0.74**	(0.028)
15 months	-0.13	0.33	0.31	-0.63	-0.17	1.00	-0.79	**0.71**	**0.65**	(0.031)
17 months	-0.08	-0.34	0.08	0.61	-0.38	-0.79	1.00	**0.38**	**0.81**	(0.024)
ASSF										
1 month	1.00	-0.91	-0.49	-0.61	-0.18	-0.66	-0.41	2.04	-0.02	(0.041)
3 months	-0.91	1.00	0.16	0.80	0.15	0.79	0.30	2.08	-0.04	(0.041)
6 months	-0.49	0.16	1.00	-0.15	-0.22	0.23	0.22	2.00	0.00	(0.041)
9 months	-0.61	0.80	-0.15	1.00	-0.23	0.43	0.57	2.27	-0.13	(0.039)
12 months	-0.18	0.15	-0.22	-0.23	1.00	0.12	-0.39	2.26	-0.13	(0.040)
15 months	-0.66	0.79	0.23	0.43	0.12	1.00	-0.27	2.06	-0.03	(0.040)
17 months	-0.41	0.30	0.22	0.57	-0.39	-0.27	1.00	2.16	-0.08	(0.041)

In summary, cross-sectional and serial correlations commonly appear to exist in the measurement errors of both the interest rate and commodity futures models. This is not surprising as various researchers, such as Eraker (2004) and Casassus and Collin-Dufresne (2005), have shown that various EAM based asset pricing models consistently overprice or underprice certain contracts. In particular, this can be a consequence of serial correlation.

3.2 Impact of cross-sectional and serial correlations on parameter estimates

In order to study how contemporaneous cross-sectional and serial measurement error correlations affect the parameter estimates, we perform Monte Carlo experiments for an EAM. Because a maximization routine must be invoked on each sample path of the Monte Carlo experiment, the computational load is heavy. We thus estimate the two factor *Gibson-Schwartz* (GS) commodity model in the simulation, and only three contracts are simulated, namely, 1, 9 and 17 months futures. Another reason for using this model is that one of the latent factors follows a mean-reverting Ornstein-Uhlenbeck process while the other follows a Brownian motion, so that we can study the behavior of these two different processes in a single model.

The simulated data were constructed as follows. We first simulate the latent factor processes with specified parameters using the Euler discretization scheme with a weekly time step for the transition equation (9), and then we calculate from (6) the 1, 9, 17 month futures prices. Cross-sectionally and serially correlated measurement errors are generated from

$$\boldsymbol{\varepsilon}_t = \Pi\varepsilon_{t-1} + \boldsymbol{u}_t, \tag{11}$$

where Π is a 3×3 matrix and $\boldsymbol{u}_t$ is a 3×1 random vector distributed joint normally with mean 0 and covariance matrix Γ. The matrices Γ and Π are specified as

$$\Gamma := \sigma_u^2 \begin{pmatrix} 1 & -\rho_c & \rho_c \\ -\rho_c & 1 & -\rho_c \\ \rho_c & -\rho_c & 1 \end{pmatrix} \quad \Pi := \begin{pmatrix} \rho_s & 0 & 0 \\ 0 & \rho_s & 0 \\ 0 & 0 & \rho_s \end{pmatrix}, \tag{12}$$

where ρ_c $(-1 \leq \rho_c \leq 1)$, ρ_s $(-1 \leq \rho_s \leq 1)$ and $\sigma_u := 0.01$ are constants[10]. From (12) we assume that the short and long futures measurement errors tend to move similarly but opposite to that of the middle (9 month) future. This allows for the effects of futures curves in backwardation or contango with possible kinks. We also assume that all three contract measurement errors follow a first order autoregressive structure with the same mean-reversion speed ρ_s.

We divide our measurement error simulation experiments into three groups. In the *first* group, we assume no serial correlation (i.e. $\Pi := 0$) but that there is contemporaneous *cross-sectional correlation*, with the correlation coefficient ρ_c set to 0.1, 0.5 or 0.9. In the *second* group, we assume no cross-sectional correlation (i.e. $\Gamma = 0$), but that there is *serial correlation*, with the auto-correlation coefficient ρ_s set to 0.1, 0.5 or 0.9. In the *third* group, we assume *both serial and cross-sectional correlation* with the pair (ρ_c, ρ_s) set to $(0.9, 0.9)$, $(0.5, 0.9)$ or $(0.9, 0.5)$. For each

[10] Note for different ρ_c we must check to see that the covariance matrix Γ is positive definite.

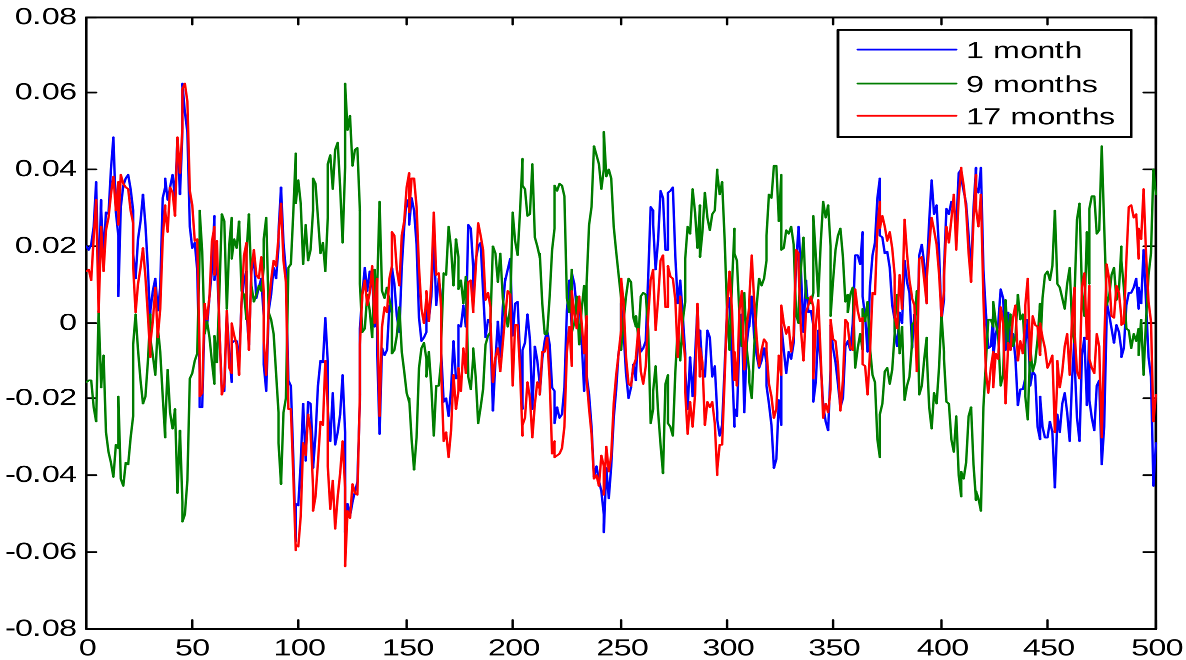

Figure 4: **The simulated measurement errors, with** $\rho_s = 0.9$**,** $\rho_c = 0.9$.

experiment we conduct 200 replications for a sample size of 500 weekly observations. Figure 4 shows a 500 observation sample path when $\rho_s := 0.9$, $\rho_c := 0.9$.

After obtaining the simulated data for each experiment, we estimate the GS model parameters for each of the 200 sample paths assuming the measurement errors covariance matrix $R := \sigma_\varepsilon I_3$. The parameters are estimated using the EM algorithm with the likelihood calculated at each iteration using the Kalman filter. The hypothetical true parameters and several descriptive statistics of their Monte Carlo estimates are reported in Tables 8, 9 and 10.

Based on the simulation results for Group 1 we see that, although the existence of contemporaneously correlated errors violates the standard SSF assumptions, the model parameter estimates are still reasonable. The mean and median of the simulated estimator distributions are quite close to the true model parameters, and their standard deviation is also relatively small. The estimation bias decreases with the cross-sectional correlation coefficient ρ_c. Note that the risk premia estimates for λ_1 and λ_2 have relatively larger standard deviations than the other parameters, which is

Table 8: **Parameter estimates from simulated data for Group 1**

Case 1: $\rho_c = 0.9$, $\rho_s = 0$								
	True value	median	mean	std dev	quantiles 5%, 25%, 75%, 95%			
k	1.00	1.031	1.030	0.048	0.956	0.996	1.060	1.106
σ_1	0.3	0.282	0.283	0.013	0.262	0.275	0.292	0.305
σ_2	0.3	0.267	0.269	0.016	0.240	0.259	0.280	0.295
λ_1	0.2	0.191	0.189	0.082	0.034	0.146	0.234	0.325
λ_2	0.1	0.093	0.092	0.091	-0.051	0.032	0.134	0.262
ρ	0.9	0.927	0.937	0.103	0.908	0.919	0.934	0.944
θ	0.1	0.104	0.104	0.009	0.093	0.100	0.109	0.114
Case 2: $\rho_c = 0.5$, $\rho_s = 0$								
k	1.00	1.013	1.019	0.043	0.943	0.994	1.049	1.094
σ_1	0.3	0.291	0.290	0.011	0.271	0.283	0.298	0.307
σ_2	0.3	0.279	0.279	0.014	0.253	0.271	0.287	0.302
λ_1	0.2	0.197	0.192	0.089	0.040	0.144	0.229	0.342
λ_2	0.1	0.097	0.091	0.092	-0.067	0.034	0.136	0.229
ρ	0.9	0.920	0.924	0.071	0.900	0.912	0.927	0.936
θ	0.1	0.103	0.103	0.006	0.094	0.099	0.106	0.112
Case 3: $\rho_c = 0.1$, $\rho_s = 0$								
k	1.00	1.009	1.008	0.041	0.935	0.983	1.033	1.078
σ_1	0.3	0.297	0.298	0.010	0.282	0.289	0.305	0.313
σ_2	0.3	0.294	0.294	0.015	0.267	0.283	0.304	0.317
λ_1	0.2	0.194	0.196	0.077	0.072	0.168	0.225	0.328
λ_2	0.1	0.101	0.098	0.075	-0.034	0.072	0.128	0.225
ρ	0.9	0.907	0.905	0.012	0.884	0.897	0.914	0.924
θ	0.1	0.100	0.100	0.005	0.092	0.097	0.104	0.108

Table 9: **Parameter estimates from simulated data for Group 2**

Case 1: $\rho_c = 0$, $\rho_s = 0.9$								
	True value	median	mean	std dev	quantiles 5%, 25%, 75%, 95%			
k	1.00	0.974	0.997	0.246	0.634	0.802	1.158	1.429
σ_1	0.3	0.271	0.272	0.012	0.253	0.263	0.281	0.293
σ_2	0.3	0.266	0.272	0.042	0.206	0.243	0.299	0.348
λ_1	0.2	0.199	0.194	0.092	0.032	0.132	0.252	0.336
λ_2	0.1	0.087	0.093	0.102	-0.085	0.029	0.157	0.265
ρ	0.9	0.901	0.902	0.015	0.873	0.893	0.912	0.927
θ	0.1	0.107	0.104	0.031	0.058	0.085	0.124	0.154
Case 2: $\rho_c = 0$, $\rho_s = 0.5$								
k	1.00	1.002	0.999	0.065	0.887	0.951	1.042	1.108
σ_1	0.3	0.293	0.293	0.010	0.276	0.286	0.299	0.314
σ_2	0.3	0.291	0.291	0.017	0.263	0.279	0.302	0.316
λ_1	0.2	0.205	0.210	0.100	0.042	0.156	0.264	0.381
λ_2	0.1	0.108	0.118	0.096	-0.045	0.070	0.173	0.274
ρ	0.9	0.902	0.902	0.012	0.884	0.893	0.911	0.921
θ	0.1	0.101	0.102	0.007	0.090	0.097	0.107	0.112
Case 3: $\rho_c = 0$, $\rho_s = 0.1$								
k	1.00	1.000	0.999	0.035	0.935	0.978	1.021	1.056
σ_1	0.3	0.297	0.298	0.010	0.280	0.291	0.305	0.317
σ_2	0.3	0.298	0.298	0.014	0.271	0.288	0.306	0.321
λ_1	0.2	0.198	0.195	0.089	0.022	0.154	0.234	0.349
λ_2	0.1	0.095	0.094	0.087	-0.053	0.049	0.138	0.235
ρ	0.9	0.899	0.899	0.011	0.883	0.891	0.906	0.919
θ	0.1	0.100	0.100	0.005	0.092	0.097	0.104	0.108

Table 10: **Parameter estimates from simulated data for Group 3**

Case 1: $\rho_c = 0.9$, $\rho_s = 0.9$								
	True value	median	mean	std dev	quantiles 5%, 25%, 75%, 95%			
k	1.00	1.113	1.177	0.513	0.486	0.806	1.448	1.985
σ_1	0.3	0.252	0.253	0.017	0.229	0.239	0.263	0.281
σ_2	0.3	0.259	0.274	0.093	0.169	0.207	0.313	0.421
λ_1	0.2	0.186	0.194	0.115	-0.017	0.124	0.270	0.384
λ_2	0.1	0.093	0.103	0.164	-0.131	-0.008	0.183	0.361
ρ	0.9	0.920	0.918	0.016	0.891	0.906	0.930	0.943
θ	0.1	0.119	0.117	0.071	0.020	0.087	0.150	0.183
Case 2: $\rho_c = 0.9$, $\rho_s = 0.5$								
k	1.00	1.035	1.058	0.103	0.916	0.994	1.107	1.246
σ_1	0.3	0.277	0.278	0.012	0.256	0.269	0.285	0.300
σ_2	0.3	0.266	0.268	0.022	0.237	0.254	0.281	0.301
λ_1	0.2	0.197	0.195	0.100	0.034	0.136	0.246	0.360
λ_2	0.1	0.100	0.100	0.106	-0.080	0.033	0.161	0.280
ρ	0.9	0.925	0.944	0.143	0.901	0.917	0.934	0.945
θ	0.1	0.107	0.107	0.011	0.090	0.100	0.114	0.123
Case 3: $\rho_c = 0.5$, $\rho_s = 0.9$								
k	1.00	1.070	1.163	0.496	0.512	0.830	1.403	2.086
σ_1	0.3	0.259	0.258	0.015	0.233	0.248	0.267	0.283
σ_2	0.3	0.256	0.280	0.091	0.184	0.223	0.314	0.439
λ_1	0.2	0.200	0.199	0.103	0.023	0.131	0.259	0.371
λ_2	0.1	0.091	0.087	0.158	-0.122	0.013	0.163	0.300
ρ	0.9	0.911	0.911	0.017	0.880	0.900	0.923	0.935
θ	0.1	0.116	0.104	0.105	0.036	0.093	0.139	0.176

consistent with risk premia estimates in the literature on commodity futures, for example, those in Schwartz (1997) and Schwartz and Smith (2000).

The results from the Group 2 experiments show that model parameter estimation becomes more inaccurate as the serial correlation increases. A comparison of Case 1 in Groups 1 and 2 shows that large serial correlation coefficient ρ_s *does* influence the parameter estimates. For Case 1 of Group 2, the serial correlation mainly influences the estimates of the mean-reverting convenience yield process parameters, i.e. the mean-reverting speed k, the long-run mean θ and the volatility σ_2, by increasing their standard deviations, although the mean and median estimates are still adequate. Hence, θ, k and σ_2 cannot be precisely estimated in this case. For instance, in Case 1, the standard deviation of k increases fivefold from about 0.05 in Group 1 to 0.25 in Group 2, from 0.009 to 0.031 for θ and from 0.015 to 0.042 for σ_2. This is easy to understand, in that the basic SSF estimation algorithm cannot tell whether the mean-reversion of futures prices (about an exponential trend) comes from the measurement errors or from the true convenience yield process.

In the Group 3 experiments, we see that the presence of the cross-sectional errors amplifies the autocorrelation estimation bias. For instance, in Cases 1 and 3, there is more than a 50% probability that the estimated mean-reversion speed $\hat{k}$ is *not* within the (0.81, 1.45) interval whose end points correspond respectively to 5.7 and 10.3 month half lives. More importantly, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ also have a large estimation bias which makes the basic SSF estimates unacceptable. In Case 2, although the cross-sectional correlation is large (0.9), the relatively small serial correlation ρ_s allows the basic parameter estimation procedure to perform reasonably well.

Therefore, the main issue regarding measurement errors is *serial correlation*, which primarily affects the parameters in mean-reverting processes (i.e., the convenience yield process in the GS model) and also the other parameters when the errors are contemporaneously cross-sectionally correlated as well. We advance a solution to this problem in the next section.

4 The Augmented State Space Form

Because contemporaneous cross-sectional and, particularly, serial measurement error correlations cause substantial imprecision in parameter estimation, we propose in this section a new SSF – based on the simple observation that the measurement error process is also *unobservable* – to address the problem.

We define the new state vector $\boldsymbol{X}$ in terms of the $\boldsymbol{Y}$ and $\boldsymbol{\varepsilon}$, i.e., $\boldsymbol{X}_t := \begin{pmatrix} \boldsymbol{Y}_t \\ \boldsymbol{\varepsilon}_t \end{pmatrix}$, and assume that $\boldsymbol{\varepsilon}_t$ follows a first order[11] vector auto-regressive process $\boldsymbol{\varepsilon}_t = \Pi\varepsilon_{t-1} + \boldsymbol{u}_t$, as in (11). Thus, the state space equations (9) and (10) become

$$\boldsymbol{X}_t = f + GX_{t-1} + \boldsymbol{w}_t \tag{13}$$

$$\boldsymbol{\mathcal{Z}}_t = \mathcal{A} + \mathcal{C}\boldsymbol{X}_t, \tag{14}$$

where $f := \begin{pmatrix} d \\ 0 \end{pmatrix}$ is a $(N+M) \times 1$ vector, $G := \begin{pmatrix} \Phi & 0 \\ 0 & \Pi \end{pmatrix}$ is an $(N+M) \times (N+M)$ matrix, $\mathcal{C} := \begin{pmatrix} \mathcal{B} & I_M \end{pmatrix}$ is an $M \times (N+M)$ matrix and $\boldsymbol{w}_t := \begin{pmatrix} \boldsymbol{\eta}_t \\ \boldsymbol{u}_t \end{pmatrix}$ is an $(N+M) \times 1$ measurement error random vector with a joint normal distribution with zero mean and covariance matrix

$$\Xi := \begin{pmatrix} \Psi & 0 \\ 0 & \Gamma \end{pmatrix},$$

where Ψ and Γ are respectively the covariance matrices for $\boldsymbol{\eta}_t$ and $\boldsymbol{u}_t$. The new state-space equations (13) and (14) allow the measurement errors $\boldsymbol{\varepsilon}_t$ to be both serially and cross-sectionally correlated. We term this SSF specification the *augmented state space form* (ASSF).

Obviously the ASSF nests the basic SSF, for which $\Pi = 0$ and Γ is a diagonal matrix. Note that in the ASSF the pricing error $[\boldsymbol{\varepsilon}_t]_i$ can be considered to be an *idiosyncratic* factor that only affects the i^{th} contract. By contrast, from (6) and (14) we see that the latent factors $\boldsymbol{Y}_t$ determine

[11] Of course, at the expense of more parameters to estimate, any ARMA or ARIMA measurement error process could be specified and the same idea employed.

(or price) *all* contracts $\boldsymbol{\mathcal{Z}}_t$ at time t and thus are *common* factors. To avoid confusion when we discuss ASSF factors in the sequel, we refer only to the common factors $\boldsymbol{Y}_t$. In the next two sections we present estimation results using both simulated and actual market data.

4.1 Simulated data estimates

The measurement errors are assumed to be specified by (11) with $\Pi := \rho_s I_3$ and $\Gamma := \sigma_u \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$. Tables 11 and 12 show the results. Since we have seen that cross-sectional error correlations minimally affect the model parameter estimates, we omit Group 1 in the simulated dataset and use only Groups 2 and 3. We see from Table 11 that relative to Table 9 the ASSF provides reasonably good estimates for the GS model for Group 2, as reflected in the relatively smaller values for the standard deviations of $\hat{k}$, $\hat{\theta}$ and $\hat{\sigma}_2$. For Group 3, we also see in Table 12 relatively smaller standard deviations for nearly all the parameters than in Table 10. More importantly, the estimates of σ_1 and σ_2 are not biased. Therefore, we may conclude that the augmented SSF is potentially better than the basic SSF for estimating EAM parameters.

4.2 Market data estimates

We now re-estimate the one, two and three factor commodity models using the market data of Section 3.1. For the measurement error structure we assume as above $\Pi := \rho_s I_3$ and $\Gamma := \sigma_u \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$ for the one and two factor models and use 1, 9, 17 month futures prices to estimate the models' parameters.

Table 11: **ASSF parameter estimates for Group 2**

Case 1: $\rho_c = 0$, $\rho_s = 0.9$								
	True value	median	mean	std dev	quantiles 5%, 25%, 75%, 95%			
k	1.00	1.010	1.013	0.130	0.809	0.910	1.090	1.250
σ_1	0.3	0.300	0.300	0.010	0.283	0.292	0.306	0.316
σ_2	0.3	0.302	0.303	0.026	0.262	0.283	0.315	0.349
λ_1	0.2	0.208	0.202	0.095	0.042	0.135	0.273	0.350
λ_2	0.1	0.100	0.105	0.105	-0.069	0.026	0.175	0.284
ρ	0.9	0.900	0.900	0.011	0.883	0.893	0.907	0.917
θ	0.1	0.100	0.097	0.024	0.057	0.080	0.114	0.136
Case 2: $\rho_c = 0$, $\rho_s = 0.5$								
k	1.00	1.005	1.002	0.064	0.889	0.953	1.047	1.099
σ_1	0.3	0.299	0.299	0.010	0.283	0.292	0.305	0.319
σ_2	0.3	0.299	0.300	0.017	0.271	0.288	0.311	0.326
λ_1	0.2	0.212	0.211	0.106	0.043	0.126	0.276	0.381
λ_2	0.1	0.120	0.118	0.103	-0.045	0.042	0.185	0.288
ρ	0.9	0.899	0.899	0.011	0.881	0.892	0.907	0.920
θ	0.1	0.100	0.100	0.007	0.088	0.095	0.105	0.111
Case 3: $\rho_c = 0$, $\rho_s = 0.1$								
k	1.00	1.000	0.999	0.034	0.938	0.978	1.021	1.057
σ_1	0.3	0.298	0.299	0.010	0.281	0.292	0.306	0.318
σ_2	0.3	0.299	0.299	0.014	0.274	0.289	0.308	0.321
λ_1	0.2	0.194	0.195	0.100	0.014	0.134	0.261	0.349
λ_2	0.1	0.089	0.095	0.097	-0.060	0.033	0.162	0.244
ρ	0.9	0.899	0.899	0.011	0.882	0.891	0.907	0.920
θ	0.1	0.100	0.100	0.005	0.092	0.097	0.103	0.107

Table 12: **ASSF parameter estimates for Group 3**

Case 1: $\rho_c = 0.9$, $\rho_s = 0.9$								
	True value	median	mean	std dev	quantiles 5%, 25%, 75%, 95%			
k	1.00	1.051	1.089	0.220	0.786	0.939	1.207	1.508
σ_1	0.3	0.299	0.299	0.012	0.280	0.290	0.307	0.319
σ_2	0.3	0.302	0.305	0.039	0.246	0.278	0.326	0.373
λ_1	0.2	0.207	0.210	0.111	0.014	0.140	0.283	0.399
λ_2	0.1	0.111	0.120	0.135	-0.079	0.027	0.219	0.360
ρ	0.9	0.910	0.913	0.018	0.889	0.900	0.923	0.946
θ	0.1	0.103	0.099	0.038	0.033	0.075	0.123	0.157
Case 2: $\rho_c = 0.9$, $\rho_s = 0.5$								
k	1.00	1.018	1.030	0.088	0.903	0.973	1.089	1.180
σ_1	0.3	0.299	0.299	0.011	0.280	0.292	0.306	0.319
σ_2	0.3	0.300	0.302	0.020	0.272	0.286	0.316	0.333
λ_1	0.2	0.199	0.199	0.105	0.039	0.126	0.266	0.370
λ_2	0.1	0.093	0.102	0.108	-0.072	0.031	0.176	0.278
ρ	0.9	0.902	0.902	0.012	0.884	0.895	0.909	0.922
θ	0.1	0.102	0.101	0.009	0.085	0.095	0.107	0.115
Case 3: $\rho_c = 0.5$, $\rho_s = 0.9$								
k	1.00	1.060	1.102	0.222	0.808	0.933	1.237	1.553
σ_1	0.3	0.298	0.299	0.010	0.282	0.292	0.306	0.315
σ_2	0.3	0.304	0.308	0.034	0.260	0.280	0.327	0.370
λ_1	0.2	0.215	0.214	0.101	0.045	0.142	0.277	0.386
λ_2	0.1	0.115	0.115	0.125	-0.084	0.033	0.196	0.322
ρ	0.9	0.909	0.911	0.023	0.876	0.894	0.926	0.950
θ	0.1	0.104	0.100	0.033	0.040	0.081	0.119	0.154

For the three factor model, we assume $\Pi := \rho_s I_7$ and a covariance matrix of the form

$$\Gamma := \begin{pmatrix} \sigma_{u,1}^2 & \cdots & \rho_{17}\sigma_{u,1}\sigma_{u,7} \\ \vdots & \ddots & \vdots \\ \rho_{17}\sigma_{u,1}\sigma_{u,7} & \cdots & \sigma_{u,7}^2 \end{pmatrix}.$$

Because this general form of Γ has too many (21) correlation parameters ρ_{ij}, we need to simply its structure. While it might seem expedient to assume that the correlations of all pairs are identical, one month futures prices show behaviour different from that of longer maturity contracts (Dempster *et al.*, 2009). Thus the correlation structure is specified as

$$\rho_{i,j} = \begin{cases} \rho_1 & (i \text{ or } j = 1, i \neq j) \\ \rho_2 & (i \geq 2, j \geq 2, i \neq j). \end{cases}$$

Tables 13 to 15 show the estimation results for the three commodity models separately. We see that large differences exist between the basic SSF and ASSF estimates for the one-factor model. Similarly, significant differences in the parameter estimates are also present in the two and three factor models, especially for the parameter k in the two-factor model and for its counterparts k_x and k_y in the three-factor model. These estimates increase by 22%, 23% and 35% respectively in the ASSF over those of the basic SSF. The estimates of σ_2 in the two-factor model and those of σ_x and σ_y in the three factor model are also significantly different in the two SSFs. Because the augmented SSF nests the basic SSF[12], the likelihood ratio (LR) test is sufficient to indicate which SSF is indeed statistically better and the test statistics indicate that the augmented SSF is significantly better than the basic SSF.[13] Figure 5 plots the evolution of the residual $\hat{u}_t$s for the two-factor GS model, while the statistics for the one, two and three factor models are summarized in Tables 5 to 7 in Section 3.1. From the DW tests and the serial correlation coefficients, we see

[12]They are equal for the one and two factor models when $\rho_s = \rho_{12} = \rho_{13} = \rho_{23} = 0$, and for the three-factor model when $\rho_s = \rho_1 = \rho_2 = 0$.

[13]The statistics for the LR tests greatly exceed the 1% significant levels of the χ^2 distribution with 3 and 4 degrees of freedom (11.3 and 13.3) respectively.

Table 13: **One factor model estimates for the basic and augmented SSFs applied to oil futures**

	Basic SSF	ASSF
k	0.088 (0.0043)	1.018 (0.0521)
σ	0.167 (0.0092)	0.300 (0.007)
λ	0.417 (0.0433)	0.735 (0.198)
α	2.368 (0.0450)	0.485 (0.065)
σ_ε (or σ_u)	0.060 (0.0114)	0.021 (0.0049)
ρ_s	–	0.990 (0.0032)
ρ_{12}	–	0.765 (0.0294)
ρ_{13}	–	0.812 (0.0480)
ρ_{23}	–	0.989 (0.0053)
Loglikelihood	2244	4836
LR Stat (*ASSF* vs. *Basic SSF*)	**5184**	

Note that quantities in brackets denote standard deviations, σ_ε is in the basic SSF and σ_u in the ASSF. Boldface denotes significant at the 1% confidence level in the likelihood ratio test (Chi-squared distribution with 4 degrees of freedom) of the augmented model.

that the residual measurement error $\widehat{\boldsymbol{u}}_t$s are not serially correlated. Thus, the estimation results are consistent with the assumptions of the model.

For the three factor interest rate model, we use the same measurement error auto-correlation structure as for the three factor commodity model. For the cross-sectional correlation structure, in order to capture the impact of maturity on correlation structure, we classify the bonds into three categories according to the time to maturity: short-term (3,6,12 months), mid-term (24,36,48 months) and long-term (60,84,120 months) bonds. We assume that the cross-sectional correlations within the same category are identical, which is denoted as ρ_1; correlations between different categories are respectively denoted as ρ_2, ρ_3 and ρ_4, which are the ones between short- and mid-term, between mid- and long- term and between short- and long-term bonds, accordingly. For the generalized Vasicek model, the estimates in Table 16 for both k_s and k_l increase by 60%, and that for k_r decreases 35%, for the ASSF relative to those for the basic SSF. The estimates of σ_1 and σ_3 are significantly different for the two SSFs. Similarly, for the $\mathbb{A}_{1,DS}$ and $\mathbb{A}_{2,DS}$ models, the estimates for the ASSF (shown in Tables 17 and 18) are quite different with those for the basic SSF. For all three models, the statistics of the residual $\widehat{\boldsymbol{u}}_t$s for the ASSF are summarized in Tables 2 to 4 of Section 3.1. These do not show large positive autocorrelation as in the basic SSF, but

Table 14: **Two factor model estimates for the basic and augmented SSFs applied to oil futures**

	Basic SSF		ASSF	
k	1.123	(0.032)	1.369	(0.052)
σ_1	0.339	(0.011)	0.357	(0.011)
σ_2	0.334	(0.014)	0.397	(0.020)
λ_1	0.302	(0.101)	0.292	(0.113)
λ_2	0.142	(0.100)	0.175	(0.138)
ρ	0.924	(0.0086)	0.924	(0.007)
θ	-0.004	(0.0052)	0.006	(0.016)
σ_ε (or σ_u)	0.012	(0.0025)	0.006	(0.000)
ρ_s	–		0.953	(0.013)
ρ_{12}	–		-0.980	(0.002)
ρ_{13}	–		-0.824	(0.002)
ρ_{23}	–		0.806	(0.003)
Loglikelihood	4094		4875	
LR Stat (*ASSF* vs. *Basic SSF*)	**1562**			

they do exhibit small negative autocorrelations which is likely due to the rather crude specification of the contemporaneous cross-sectional correlation structure of the measurement errors. However, for the most part of Durbin-Watson statistics are *not* significant particularly for the $\mathbb{A}_{1,DS}(3)$ model. From the likelihood ratio test, the ASSF is *very* significantly better than the basic SSF for all three models. Again, for the $\mathbb{A}_{1,DS}(3)$ model, the improvement is an order of magnitude in the likelihood ratio test statistic over the other two models.

In summary, for both the actual market and simulated data, we can see that the ASSF performs reasonably well in estimating EAMs and much better than the basic SSF.

5 Conclusion

This paper focusses on how measurement errors affect parameter estimates in exponential affine factor models. Investigating measurement errors in three three-factor yield curve models, and one, two and three factor commodity futures models, we find that residual measurement errors are not independently and identically distributed but rather show strong contemporaneous cross-sectional and serial correlations. This is inconsistent with the usual iid assumptions in many studies employ-

Table 15: **Three factor model estimates for the basic and augmented SSFs applied to oil futures**

	Basic SSF		ASSF	
k_x	2.264	(0.0773)	2.786	(0.0640)
k_y	0.637	(0.0362)	0.865	(0.0176)
u	0.023	(0.0053)	0.002	(0.0057)
σ_x	0.250	(0.0455)	0.221	(0.0055)
σ_y	0.321	(0.0116)	0.308	(0.0092)
σ_p	0.172	(0.0065)	0.156	(0.0045)
λ_x	-0.020	(0.0317)	-0.009	(0.0229)
λ_y	0.425	(0.1367)	0.536	(0.0042)
λ_p	0.095	(0.0010)	0.236	(0.0010)
ρ_{xy}	-0.254	(0.0650)	-0.207	(0.0288)
ρ_{xp}	0.322	(0.0463)	0.271	(0.0330)
ρ_{yp}	-0.439	(0.0518)	-0.145	(0.0181)
ρ_s	–		0.828	(0.0117)
ρ_1	–		-0.403	(0.078)
ρ_2	–		-0.005	(0.085)
σ_1(or $\sigma_{u,1}$)	0.0173	(0.0042)	0.0089	(0.0003)
σ_2(or $\sigma_{u,2}$)	0.0000	(0.0000)	0.0001	(0.0000)
σ_3(or $\sigma_{u,3}$)	0.0029	(0.0007)	0.0015	(0.0001)
σ_4(or $\sigma_{u,4}$)	0.0006	(0.0005)	0.0004	(0.0001)
σ_5(or $\sigma_{u,5}$)	0.0014	(0.0004)	0.0010	(0.0001)
σ_6(or $\sigma_{u,6}$)	0.0003	(0.0003)	0.0004	(0.0001)
σ_7(or $\sigma_{u,7}$)	0.0018	(0.0005)	0.0009	(0.0001)
Loglikelihood	15347		16668	
LR Stat (*ASSF* vs. *Basic SSF*)	**2642**			

Table 16: **Three factor generalized Vasicek interest rate model estimates for the basic and augmented SSFs**

	Basic SSF	ASSF
k_s	0.8250 (0.0578)	1.3363 (0.1195)
k_l	0.0142 (0.0015)	0.0226 (0.0020)
k_r	1.1815 (0.1022)	0.7699 (0.0519)
σ_1	0.0210 (0.0018)	0.0308 (0.0033)
σ_2	0.0101 (0.0003)	0.0110 (0.0003)
σ_3	0.0104 (0.0003)	0.0077 (0.0002)
u_s	-0.1899 (0.0332)	0.0314 (0.0078)
u_l	0.0049 (0.0000)	0.0016 (0.0002)
γ_1	-0.4003 (0.2048)	-0.3542 (0.2054)
γ_2	-0.0580 (0.0788)	0.0386 (0.0961)
γ_3	-0.7120 (0.2219)	-0.8531 (0.2196)
ρ_{12}	0.2204 (0.0533)	0.2613 (0.0395)
ρ_{13}	-0.4080 (0.0323)	-0.1395 (0.0384)
ρ_{23}	0.0190 (0.0351)	0.2673 (0.0328)
ρ_s	–	0.7667 (0.0328)
ρ_1	–	0.9253 (0.0058)
ρ_2	–	0.5517 (0.0299)
ρ_3	–	0.8671 (0.0150)
ρ_4	–	0.5666 (0.0302)
σ_1(or $\sigma_{u,1}$)	0.00097 (0.00021)	0.00050 (0.00018)
σ_2(or $\sigma_{u,2}$)	0.00001 (0.00000)	0.00074 (0.00020)
σ_3(or $\sigma_{u,3}$)	0.00072 (0.00016)	0.00091 (0.00022)
σ_4(or $\sigma_{u,4}$)	0.00034 (0.00008)	0.00048 (0.00014)
σ_5(or $\sigma_{u,5}$)	0.00008 (0.00004)	0.00040 (0.00011)
σ_6(or $\sigma_{u,6}$)	0.00015 (0.00005)	0.00032 (0.00010)
σ_7(or $\sigma_{u,7}$)	0.00021 (0.00007)	0.00033 (0.00011)
σ_8(or $\sigma_{u,8}$)	0.00020 (0.00009)	0.00044 (0.00012)
σ_9(or $\sigma_{u,9}$)	0.00046 (0.00013)	0.00053 (0.00014)
Loglikelihood	57788	61454
LR Stat (*ASSF* vs. *Basic SSF*)	**7332**	

Table 17: **Three factor $A_{1,DS}(3)$ interest rate model estimates for the basic and augmented SSFs**

	Basic SSF	ASSF
a	0.0802 (0.0006)	0.2595 (0.0348)
k	0.3540 (0.0042)	0.2132 (0.0312)
u	0.0831 (0.0020)	0.6003 (0.18353)
η	0.0410 (0.0011)	0.0035 (0.0007)
$\overline{v}$	0.0523 (0.0021)	0.0011 (0.0001)
$\overline{\theta}$	0.2941 (0.0039)	0.0780 (0.0016)
ζ	0.0183 (0.0042)	0.0280 (0.0037)
b	-0.0132 (0.0020)	0.0026 (0.0014)
σ_{rv}	0.1606 (0.0035)	-0.1254 (0.2181)
c	0.0073 (0.0032)	-0.1519 (0.0544)
λ_1	6.8730 (3.5396)	-1.0708 (3.7815)
λ_2	-1.4138 (0.7281)	-0.1604 (0.0655)
λ_3	0.2131 (0.0038)	0.2706 (0.1304)
ρ_s	–	0.9592 (0.0046)
ρ_1	–	0.8390 (0.0125)
ρ_2	–	0.1471 (0.0580)
ρ_3	–	0.6738 (0.0340)
ρ_4	–	0.3222 (0.0415)
σ_1(or $\sigma_{u,1}$)	0.00356 (0.00006)	0.00102 (0.00028)
σ_2(or $\sigma_{u,2}$)	0.00236 (0.00004)	0.00087 (0.00025)
σ_3(or $\sigma_{u,3}$)	0.00101 (0.00012)	0.00090 (0.00024)
σ_4(or $\sigma_{u,4}$)	0.00000 (0.00000)	0.00037 (0.00011)
σ_5(or $\sigma_{u,5}$)	0.00031 (0.00001)	0.00030 (0.00009)
σ_6(or $\sigma_{u,6}$)	0.00019 (0.00001)	0.00022 (0.00009)
σ_7(or $\sigma_{u,7}$)	0.00028 (0.00002)	0.00024 (0.00007)
σ_8(or $\sigma_{u,8}$)	0.00000 (0.0000)	0.00035 (0.00008)
σ_9(or $\sigma_{u,9}$)	0.00056 (0.00003)	0.00046 (0.00010)
Loglikelihood	52624	60448
LR Stat (*ASSF* vs. *Basic SSF*)	**14568**	

Table 18: **Three factor $A_{2,DS}(3)$ interest rate model estimates for the basic and augmented SSFs**

	Basic SSF	ASSF
a	0.0224 (0.0005)	0.0353 (0.0006)
k	0.2776 (0.0092)	0.4094 (0.0070)
u	0.0429 (0.0040)	0.1458 (0.0601)
η	0.0225 (0.0001)	0.0913 (0.0052)
$\overline{v}$	0.0103 (0.0014)	0.0329 (0.0005)
$\overline{\theta}$	0.2297 (0.0099)	0.3158 (0.0003)
ζ	0.0474 (0.0009)	0.0322 (0.0013)
$k_{\theta v}$	0.1324 (0.0141)	0.1114 (0.0442)
k_{rv}	0.1298 (0.0211)	0.1379 (0.0236)
σ_{rv}	-0.0942 (0.0297)	0.0843 (0.0145)
λ_1	-0.1348 (3.3137)	-0.0011 (0.0010)
λ_2	0.2911 (0.7637)	-0.0067 (0.0014)
λ_3	0.0766 (0.2874)	-0.0018 (0.0013)
ρ_s	–	0.9292 (0.0161)
ρ_1	–	0.7197 (0.0040)
ρ_2	–	0.4014 (0.0902)
ρ_3	–	0.3107 (0.0034)
ρ_4	–	0.3016 (0.0046)
σ_1(or $\sigma_{u,1}$)	0.00387 (0.00082)	0.00259 (0.00016)
σ_2(or $\sigma_{u,2}$)	0.00265 (0.00057)	0.00188 (0.00012)
σ_3(or $\sigma_{u,3}$)	0.00129 (0.00029)	0.00120 (0.00023)
σ_4(or $\sigma_{u,4}$)	0.00027 (0.00006)	0.00029 (0.00002)
σ_5(or $\sigma_{u,5}$)	0.00009 (0.00004)	0.00000 (0.00000)
σ_6(or $\sigma_{u,6}$)	0.00014 (0.00003)	0.00017 (0.00002)
σ_7(or $\sigma_{u,7}$)	0.00021 (0.00005)	0.00019 (0.00001)
σ_8(or $\sigma_{u,8}$)	0.00009 (0.00004)	0.00026 (0.00001)
σ_9(or $\sigma_{u,9}$)	0.00039 (0.00008)	0.00040 (0.00004)
Loglikelihood	53155	57124
LR Stat (*ASSF* vs. *Basic SSF*)	**7938**	

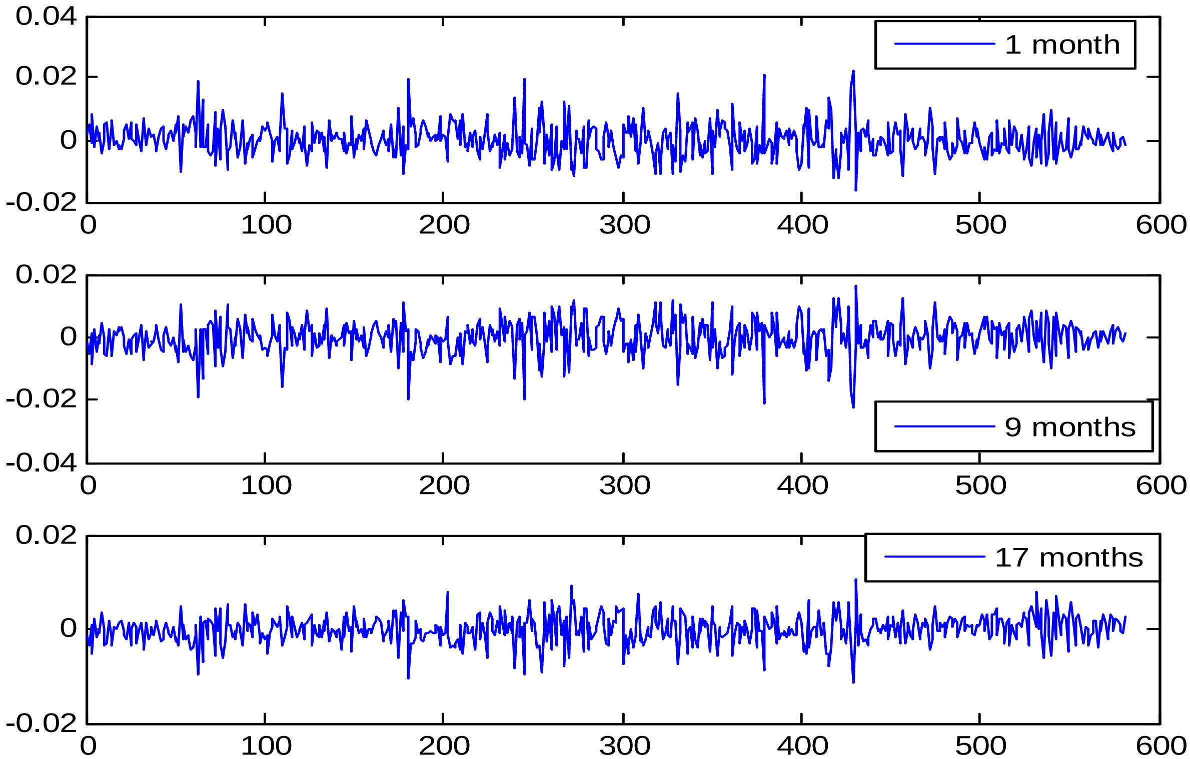

Figure 5: **The measurement errors for the augmented state space form of the GS commodity model**

ing EAMs in the fixed income and commodity literature, such as Schwartz (1997), de Jong (2000) and Chen and Scott (2003). By performing Monte Carlo simulations, we find that if no serial, but only cross-sectional correlation exists, the usual basic SSF estimation procedure performs reasonably well. However, when serial correlation exists, the basic SSF estimation procedure performs very poorly – especially regarding the estimation of the parameters of mean-reverting processes. This is because the Kalman filter estimation process cannot distinguish mean-reversion directly due to the underlying process from that arising from measurement errors. To resolve this issue, we propose an augmented SSF to replace the original and employ the Kalman filter to estimate the ASSF parameters using simulated and actual market data. The results demonstrate that the new ASSF performs much better than the original SSF in estimating EAM parameters.

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Appendixes:

A Kalman Filter Parameter Estimation

All models in this paper are calibrated using a generalized iterative *estimation and maximization* (EM) algorithm (Dempster *et al.*, 1976, 1977) based on the Kalman filter. Here we briefly describe this procedure of the Kalman filter. Notice that the Kalman filter estimates the *linear* and *Gaussian* models as shown in (15) and (16), which are the transition and measurement equations, respectively.

$$\boldsymbol{Y}_t = d + \Phi Y_{t-1} + \boldsymbol{\eta}_t \tag{15}$$

$$\boldsymbol{\mathcal{Z}}_t = \mathcal{A} + \mathcal{B} Y_t + \boldsymbol{\varepsilon}_t. \tag{16}$$

For non-Gaussian models, the so-called *extended* Kalman filter should be used. It first "normalizes" the model by finding the first two moments for the latent factors from their conditional density. Thus, non-Gaussian models can be rewritten in the state-space form of (15) and (16) and the standard Kalman filter applied for model estimation. We refer readers to De Jong (2000), Chen and Scott (2003), Duffee and Stanton (2004) for the detailed normalization procedure and the use of extended Kalman filters.

Given values for all parameters, after determining initial values for the latent factors the Kalman filtering procedure at each time step t can be conducted in three steps – prediction, likelihood calculation and updating.

Initial conditions

Initial values of state vector $\widehat{Y}_0$ and its variance $\widehat{P}_0$ must be specified to use the Kalman filter. When the state variable process is stationary ergodic, the long-run mean and variance are usually taken for $\widehat{Y}_0$ and $\widehat{P}_0$ respectively. If the state variable process is not stationary, the initial value $\widehat{Y}_0$ is usually taken to be an unknown parameter of the model (to be estimated) and an arbitrary large value (say 10^5) is used for $\widehat{P}_0$ (see Harvey 1989).

Prediction

$$
\begin{aligned}
Y_{t|t-1} &= d + \Phi \widehat{Y}_{t-1} && (17)\\
P_{t|t-1} &= \Phi \widehat{Y}_{t-1} \Phi' + \Omega_t. && (18)
\end{aligned}
$$

Incremental likelihood calculation

$$
\begin{aligned}
e_t &= \mathcal{Z}_t - \mathcal{A} - \mathcal{B} Y_{t|t-1} \\
V_t &= \mathcal{B} P_{t|t-1} \mathcal{B}' + R && (19)\\
\ln L_t &= -\frac{1}{2} \ln |V_t| - \frac{1}{2} e_t' V_t^{-1} e_t,
\end{aligned}
$$

where L_t denotes the likelihood contribution at time t.

Updating

$$
\begin{aligned}
K_t &= P_{t|t-1} \mathcal{B}' V_t^{-1} \\
\widehat{Y}_t &= Y_{t|t-1} + K_t e_t && (20)\\
\widehat{P}_t &= (I - K_t \mathcal{B}) P_{t|t-1}
\end{aligned}
$$

Note that the total log-likelihood L for the whole sample is calculated as

$$
L(\theta) = \sum_{t=1}^{T} \ln L_t(\theta),
$$

where θ denotes the full set of EAM parameters. Using a suitable global optimization algorithm the optimal estimate $\hat{\theta}$ is found for the current estimate $\hat{Y}$ of the factor process evolution.

Set $\theta := \hat{\theta}$ and repeat the estimation and maximization steps until convergence for $\hat{\theta}$ (suitably specified) is achieved.

B Three Factor Yield Curve Models

1) The Generalized Vasicek model

To do a cross-market robustness check, we estimate a three factor interest rate term structure model specified in the risk neutral measuqre as

$$\begin{aligned} d\boldsymbol{s}_t &= k_s(\theta_s - s_t)dt + \sigma_1 d\boldsymbol{W}_1^Q \\ d\boldsymbol{l}_t &= k_l(\theta_l - l_t)dt + \sigma_2 d\boldsymbol{W}_2^Q \\ d\boldsymbol{r}_t &= k_r(s_t + l_t - r_t)dt + \sigma_3 d\boldsymbol{W}_3^Q \end{aligned} \tag{21}$$

$$E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_2^Q] = \rho_{12}dt, \quad E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_3^Q] = \rho_{13}dt, \quad E[d\boldsymbol{W}_2^Q d\boldsymbol{W}_3^Q] = \rho_{23}dt,$$

where $\boldsymbol{s}_t$ and $\boldsymbol{l}_t$ are assumed to represent respectively a long interest rate (or yield curve level) and the (negative) slope of the yield curve with respect to a *perceived* instantaneous short interest rate which the instantaneous short rate $\boldsymbol{r}_t$ tracks. In the market measure (21) becomes

$$\begin{aligned} d\boldsymbol{s}_t &= [k_s(\theta_s - s_t) + \sigma_1\gamma_1]dt + \sigma_1 d\boldsymbol{W}_1^P \\ d\boldsymbol{l}_t &= [k_l(\theta_l - l_t) + \sigma_2\gamma_2]dt + \sigma_2 d\boldsymbol{W}_2^P \\ d\boldsymbol{r}_t &= [k_r(s_t + l_t - r_t) + \sigma_3\gamma_3]dt + \sigma_3 d\boldsymbol{W}_3^P \end{aligned} \tag{22}$$

$$E[d\boldsymbol{W}_1^P d\boldsymbol{W}_2^P] = \rho_{12}dt, \quad E[d\boldsymbol{W}_1^P d\boldsymbol{W}_3^P] = \rho_{13}dt, \quad E[d\boldsymbol{W}_2^P d\boldsymbol{W}_3^P] = \rho_{23}dt,$$

where γ_1, γ_2 and γ_3 are risk premia.

Note that this model belongs to the $\mathbb{A}_0(3)$ class of Dai and Singleton (2000) and so zero-coupon yield and bond prices in terms of the factors can easily be obtained in closed form (see Medova *et al.*, 2006).

2) Dai-Singleton $\mathbb{A}_{1,DS}(3)$ model

We choose the preferred model $\mathbb{A}_{1,DS}(3)$ of Dai and Singleton (2000) as our $\mathbb{A}_1(3)$ model.

In the risk-neutral measure the process is specified by

$$
\begin{aligned}
d\boldsymbol{v}_t &= u\left(\overline{v} - v_t\right) dt + \eta\sqrt{v_t} d\boldsymbol{W}_1^Q \\
d\boldsymbol{\theta}_t &= a(\overline{\theta} - \theta_t)dt + \zeta d\boldsymbol{W}_2^Q + c\sqrt{v_t} d\boldsymbol{W}_3^Q \\
d\boldsymbol{r}_t &= k\left(\theta_t - r_t\right) dt + \sigma_{rv}\sqrt{v_t} d\boldsymbol{W}_1^Q + bd\boldsymbol{W}_2^Q + \sqrt{v_t} d\boldsymbol{W}_3^Q
\end{aligned}
\tag{23}
$$

$$E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_2^Q] = 0, \quad E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_3^Q] = 0, \quad E[d\boldsymbol{W}_2^Q d\boldsymbol{W}_3^Q] = 0.$$

In the market measure, the process satisfies

$$
\begin{aligned}
d\boldsymbol{v}_t &= \left[u\overline{v} + \left(\lambda_1\eta - u\right) v_t\right] dt + \eta\sqrt{v_t} d\boldsymbol{W}_1^P \\
d\boldsymbol{\theta}_t &= \left[\left(a\overline{\theta} + \zeta\lambda_2\right) + \lambda_3 c v_t - a\theta_t\right] dt + \zeta d\boldsymbol{W}_2^P + c\sqrt{v_t} d\boldsymbol{W}_3^P \\
d\boldsymbol{r}_t &= \left[b\lambda_2 + \left(\sigma_{rv}\lambda_1 + \lambda_3\right) v_t + k\theta_t - kr_t\right] dt + \sigma_{rv}\sqrt{v_t} d\boldsymbol{W}_1^P + bd\boldsymbol{W}_2^P + \sqrt{v_t} d\boldsymbol{W}_3^P
\end{aligned}
\tag{24}
$$

$$E[d\boldsymbol{W}_1^P d\boldsymbol{W}_2^P] = 0, \quad E[d\boldsymbol{W}_1^P d\boldsymbol{W}_3^P] = 0, \quad E[d\boldsymbol{W}_2^P d\boldsymbol{W}_3^P] = 0,$$

where $\lambda_1\sqrt{v_t}$, λ_2 and $\lambda_3\sqrt{v_t}$ are risk premia.

3) Dai-Singleton $\mathbb{A}_{2,DS}(3)$ *model*

We choose the preferred model $\mathbb{A}_{2,DS}(3)$ of Dai and Singleton (2000) as our $\mathbb{A}_2(3)$ model.

In the risk-neutral measure the process is specified by

$$
\begin{aligned}
d\boldsymbol{v}_t &= u\left(\overline{v} - v_t\right) dt + \eta\sqrt{v_t} d\boldsymbol{W}_1^Q \\
d\boldsymbol{\theta}_t &= a(\overline{\theta} - \theta_t)dt + k_{\theta v}\left(\overline{v} - v_t\right) dt + \zeta\sqrt{\theta_t} d\boldsymbol{W}_2^Q \\
d\boldsymbol{r}_t &= k_{rv}\left(\overline{v} - v_t\right) dt + k\left(\theta_t - r_t\right) dt + \sigma_{rv}\sqrt{v_t} d\boldsymbol{W}_1^Q + \sqrt{v_t} d\boldsymbol{W}_3^Q
\end{aligned}
\tag{25}
$$

$$E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_2^Q] = 0, \quad E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_3^Q] = 0, \quad E[d\boldsymbol{W}_2^Q d\boldsymbol{W}_3^Q] = 0.$$

In the market measure the process satisfies

$$\begin{aligned}
d\boldsymbol{v}_t &= \left[u\overline{v} + (\eta\lambda_1 - u)\, v_t\right] dt + \eta\sqrt{v_t}\, d\boldsymbol{W}_1^P \\
d\boldsymbol{\theta}_t &= \left[\left(a\overline{\theta} + k_{\theta v}\overline{v}\right) - k_{\theta v} v_t + (\zeta\lambda_2 - a)\,\theta_t\right] dt + \zeta\sqrt{\theta_t}\, d\boldsymbol{W}_2^P \\
d\boldsymbol{r}_t &= \left[k_{rv}\overline{v} + (\lambda_1\sigma_{rv} + \lambda_3 - k_{rv})\, v_t + k\theta_t - kr_t\right] dt + \sigma_{rv}\sqrt{v_t}\, d\boldsymbol{W}_1^P + \sqrt{v_t}\, d\boldsymbol{W}_3^P
\end{aligned} \tag{26}$$

$$E[d\boldsymbol{W}_1^P d\boldsymbol{W}_2^P] = 0, \quad E[d\boldsymbol{W}_1^P d\boldsymbol{W}_3^P] = 0, \quad E[d\boldsymbol{W}_2^P d\boldsymbol{W}_3^P] = 0,$$

where $\lambda_1\sqrt{v_t}$, $\lambda_2\sqrt{\theta_t}$ and $\lambda_3\sqrt{v_t}$ are risk premia.

C Commodity Futures Pricing Models

Here we set out the one, two and three factor models for commodity futures used in the paper. For brevity we ignore many details. Interested readers may refer to Schwartz (1997) and Dempster, Medova and Tang (2009) for these.

1) One Factor Model

The one factor model in the market measure is a geometric Ornstein-Uhlenbeck process given by the solution of

$$d\boldsymbol{v}_t = k(\alpha - v_t)dt + \sigma d\boldsymbol{W}^P, \tag{27}$$

where $\boldsymbol{v}_t$ is the logarithm of the spot price $\boldsymbol{S}_t$. Assuming a constant risk premium λ, the dynamics under the risk-neutral measure follow

$$d\boldsymbol{v}_t = [k(\alpha - v_t) - \lambda]dt + \sigma d\boldsymbol{W}^Q. \tag{28}$$

2) Two Factor Model

In the market measure, the two factor (Gibson-Schwartz) model is specified by

$$\begin{aligned} d\boldsymbol{v}_t &= (r - \delta_t - \frac{1}{2}\sigma_1^2 + \lambda_1)dt + \sigma_1 d\boldsymbol{W}_1^P \\ d\boldsymbol{\delta}_t &= k(\theta - \delta_t)dt + \sigma_2 d\boldsymbol{W}_2^P \end{aligned} \tag{29}$$

$$E[d\boldsymbol{W}_1^P d\boldsymbol{W}_2^P] = \rho dt, \tag{30}$$

where $\boldsymbol{v}_t$ is the logarithm of the spot price $\boldsymbol{S}_t$ and $\boldsymbol{\delta}_t$ is the convenience yield. By assuming constant risk premia, the dynamics under the risk-neutral measure follow

$$\begin{aligned} d\boldsymbol{v}_t &= (r - \delta_t - \frac{1}{2}\sigma_1^2)dt + \sigma_1 d\boldsymbol{W}_1^Q \\ d\boldsymbol{\delta}_t &= [k(\theta - \delta_t) - \lambda_2]dt + \sigma_2 d\boldsymbol{W}_2^Q \end{aligned} \tag{31}$$

$$E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_2^Q] = \rho dt. \tag{32}$$

3) Three Factor Model

In the market measure, the three factor (Dempster-Medova-Tang) model is specified by

$$\begin{aligned} d\boldsymbol{v}_t &= (r - \alpha_t - \gamma_t - \frac{1}{2}\sigma_1^2 + \lambda_1)dt + \sigma_1 d\boldsymbol{W}_1^P \\ d\boldsymbol{\alpha}_t &= k_\delta(\alpha - \delta_t)dt + \sigma_2 d\boldsymbol{W}_2^P \\ d\boldsymbol{\gamma}_t &= -k_\gamma \gamma_t dt + \sigma_3 d\boldsymbol{W}_3^P \end{aligned} \tag{33}$$

$$E[d\boldsymbol{W}_1^P d\boldsymbol{W}_2^P] = \rho_{12}dt, \quad E[d\boldsymbol{W}_1^P d\boldsymbol{W}_3^P] = \rho_{13}dt, \quad E[d\boldsymbol{W}_2^P d\boldsymbol{W}_3^P] = \rho_{23}dt, \tag{34}$$

where the convenience yield $\boldsymbol{\delta}_t$ is decomposed into two mean reverting components, i.e., $\delta_t := \alpha_t + \gamma_t$.

The dynamics under the risk-neutral measure follow

$$\begin{aligned} d\boldsymbol{v}_t &= (r - \alpha_t - \gamma_t - \frac{1}{2}\sigma_1^2)dt + \sigma_1 d\boldsymbol{W}_1^Q \\ d\boldsymbol{\alpha}_t &= [k_\delta(\alpha - \delta_t) - \lambda_2]dt + \sigma_2 d\boldsymbol{W}_2^Q \\ d\boldsymbol{\gamma}_t &= [-k_\gamma \gamma_t - \lambda_3]dt + \sigma_3 d\boldsymbol{W}_3^Q \end{aligned} \tag{35}$$

$$E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_2^Q] = \rho_{12}dt, \quad E[d\boldsymbol{W}_1^Q d\boldsymbol{W}_3^Q] = \rho_{13}dt, \quad E[d\boldsymbol{W}_2^Q d\boldsymbol{W}_3^Q] = \rho_{23}dt. \tag{36}$$

Defining $\ln(\boldsymbol{S}_t) := \boldsymbol{x}_t + \boldsymbol{y}_t + \boldsymbol{p}_t$, as shown in Dempster, Medova and Tang (2009), (33) and (35) are equivalent to the following dynamics in the market and risk-neutral measures respectively

$$\begin{aligned} d\boldsymbol{x}_t &= -k_x x_t dt + \sigma_x d\boldsymbol{W}_x^P \\ d\boldsymbol{y}_t &= -k_y y_t dt + \sigma_y d\boldsymbol{W}_y^P \\ d\boldsymbol{p}_t &= u dt + \sigma_p d\boldsymbol{W}_p^P \end{aligned} \tag{37}$$

$$E[d\boldsymbol{W}_x^P d\boldsymbol{W}_y^P] = \rho_{xy}dt, \quad E[d\boldsymbol{W}_x^P d\boldsymbol{W}_p^P] = \rho_{xp}dt, \quad E[d\boldsymbol{W}_y^P d\boldsymbol{W}_p^P] = \rho_{yp}dt \tag{38}$$

and

$$\begin{aligned} d\boldsymbol{x}_t &= k_x(-x_t - \lambda_x)dt + \sigma_x d\boldsymbol{W}_x^Q \\ d\boldsymbol{y}_t &= k_y(-y_t - \lambda_y)dt + \sigma_y d\boldsymbol{W}_y^Q \\ d\boldsymbol{p}_t &= (u - \lambda_p)dt + \sigma_p d\boldsymbol{W}_p^Q \end{aligned} \tag{39}$$

$$E[d\boldsymbol{W}_x^Q d\boldsymbol{W}_y^Q] = \rho_{xy}dt, \quad E[d\boldsymbol{W}_x^Q d\boldsymbol{W}_p^Q] = \rho_{xp}dt, \quad E[d\boldsymbol{W}_y^Q d\boldsymbol{W}_p^Q] = \rho_{yp}dt. \tag{40}$$

In this paper, we estimate the parameters of (37) and (39).

Note that the one, two and three factor models respectively belong to the $\mathbb{A}_0(N),\ N = 1, 2, 3$, classes of Dai and Singleton (2000) and futures prices in terms of the factors are easily derived.